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ON THE AVERAGE DISTANCES IN A CIRCULAR DISC*

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Abstract. Using the l^p notion of distance in the Cartesian plane, and assuming a uniform density of locations on a circular disc, we consider the resulting distance to any specified point of this domain, and we determine the first two moments of this random variable for $p = 1, 2, \infty$. We find the maxima and minima of these average distances and their ratios, hence show their almost exact proportionality over the disc. Situations motivating these results include traffic flow on a rectangular street grid in a circular city and physical design of certain computer systems in two dimensions.

1. Introduction. A common model for urban transportation is traffic flow in a circular city. Various authors (Smeed [16], Smeed and Jeffcoate [17], Fairthorne [5], Tan [18], Einhorn [4], Holroyd [11], Pearce [13]) assume a disc city with either a rectangular or a polar street grid, consider travel distances along either grid or straight-line paths, obtain double averages over both initial and final points, then essay policy conclusions for both real and hypothetical cities. However, the average distance from a fixed point has received somewhat less attention in this literature (Haight [7], Witzgall [21]). The physical design of a computer generates optimal placement problems in two dimensions which involve similar average distances from a given point (Hanan and Kurtzberg [10], Karp, McKellar and Wong [12], Wong and Chu [22]). Often the appropriate distance for such problems becomes the maximum absolute value of the coordinate differences, while uniform distributions on the relevant domains yield important results for any further analysis, and circular domains, in many cases, provide a close approximation to the optimal shapes.

Certain recent models of disc cities involve some radial dependence for the population density, but cited empirical data on such densities give no clear indication of the functional form (Pearce [13]). Moreover, a probability density, in the contexts of these models, may represent the spatial distribution of other entities, such as fires, workplaces, or accidents. Thus a uniform density offers at least a universal first approximation, and the ultimate issues may demand only broad geometrical assertions (Plattner [14]). Hence we suppose, for concreteness, that a circular city of given radius has a fine rectangular grid of streets; and we require, for simplicity, that the probability distribution of the relevant locations is uniform on the disc: we assume, in other words, that the probability measure of any Borel set is proportional to its area. We choose an arbitrary point in this circular disc, consider its l^p distance to a random location, and calculate the first two distance moments for $p = 1, 2, \infty$. This yields properties of the average distances, and their ratios, which determine maxima and minima on the disc. The almost exact proportionality of these averages is a noteworthy consequence of this work.

We introduce a system (x, y) of rectangular coordinates, and define an associated pair $(\mathbf{i}, \mathbf{j})$ of unit vectors, with the origin located at the disc center and the axes parallel to the street grid. Our unit of length will be the radius of the city, whence the disc city

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will have a representation

$$D=\{(x, y): x^2+y^2 \leqq 1\}, \tag{1.1}$$

and a location in any area $dx\,dy$ will have a probability of $dx\,dy/\pi$. From an arbitary point (u, v) in these coordinates to a random point (x, y) with uniform distribution, the l^p distance for $1 \leqq p < \infty$ is $\{|u-x|^p+|v-y|^p\}^{1/p}$, so that its nth moment, for $n = 1, 2, \cdots$, is

$$a_n^p(u, v)=\iint_D \{|u-x|^p+|v-y|^p\}^{n/p}\,dx\,dy/\pi, \tag{1.2}$$

and the l^∞ distance, as usual, is $\max(|u-x|, |v-y|)$, so that its nth moment, in the same way, is

$$a_n^\infty(u, v)=\iint_D \{\max(|u-x|, |v-y|)\}^n\,dx\,dy/\pi. \tag{1.3}$$

The obvious symmetries of (1.2) and (1.3) yield

$$a_n^p(v, u)=a_n^p(u, v)=a_n^p(|u|, |v|) \quad \text{for all } p \text{ and } n. \tag{1.4}$$

For any real number ψ we recall the vector function

$$\mathbf{e}(\psi)=\mathbf{i}\cos\psi+\mathbf{j}\sin\psi, \tag{1.5}$$

and for our generic points we define the further representations

$$\mathbf{q}=q\mathbf{e}(\phi)=u\mathbf{i}+v\mathbf{j}, \qquad \mathbf{r}=r\mathbf{e}(\theta)=x\mathbf{i}+y\mathbf{j}. \tag{1.6}$$

We use the polar coordinates for (u, v) to express the rotational average of a_n^p:

$$b_n^p(q)=(2\pi)^{-1}\int_0^{2\pi} a_n^p(q\cos\phi, q\sin\phi)\,d\phi. \tag{1.7}$$

If we rotate the point (u, v) onto the positive x-axis, then we do not change the average $a_n^2(u, v)$ in the Euclidean sense, so that

$$a_n^2(u, v)=a_n^2(q, 0)=b_n^2(q), \tag{1.8}$$

where $q^2=u^2+v^2$. However if $p \neq 2$ then $a_n^p(u, v)$ is not rotation-invariant. The variance of the l^p distance is $a_2^p(u, v)-[a_1^p(u, v)]^2$ by a standard identity. Thus we shall investigate the distance moments for $n = 1, 2$; and we can evaluate the resulting integrals for $p = 1, 2, \infty$. However if, for any real s, t, we recall the identity

$$|s+t|+|s-t|=2\max(|s|, |t|); \tag{1.9}$$

and if, in definition (1.3), we substitute the variables

$$u'=(u+v)/\sqrt{2}, \qquad v'=(u-v)/\sqrt{2}; \tag{1.10}$$

$$x'=(x+y)/\sqrt{2}, \qquad y'=(x-y)/\sqrt{2}; \tag{1.11}$$

then directly, by (1.9), we obtain the reduction

$$\begin{aligned} a_n^\infty(u, v) &= 2^{-n} \iint_D \{|u+v-x-y| + |u-v-x+y|\}^n \, dx \, dy/\pi \\ &= 2^{-n/2} \iint_D \{|u'-x'| + |v'-y'|\}^n \, dx' \, dy'/\pi = 2^{-n/2} a_n^1(u', v'). \end{aligned} \tag{1.12}$$

Hence our study of these moments requires no further mention of $a_n^\infty(u, v)$.

Moreover, our calculated average for the rectangular distance will yield the corresponding result for a nonorthogonal street grid. If $\mathbf{e}_1, \mathbf{e}_2$ are unit vectors parallel to the grid directions, then $\mathbf{k} = (\mathbf{e}_1 \times \mathbf{e}_2)/|\mathbf{e}_1 \times \mathbf{e}_2|$ is a unit vector perpendicular to the disc, and $\mathbf{e}_1^* = \mathbf{e}_2 \times \mathbf{k}$, $\mathbf{e}_2^* = \mathbf{k} \times \mathbf{e}_1$ are unit vectors orthogonal respectively to $\mathbf{e}_2, \mathbf{e}_1$. Moreover, these vectors satisfy

$$\mathbf{e}_1 \cdot \mathbf{e}_1^* = \mathbf{e}_2 \cdot \mathbf{e}_2^* = |\mathbf{e}_1 \times \mathbf{e}_2| = \sin \alpha,$$

where the grid directions define the angle α, and the average distance becomes

$$\iint_D |\mathbf{e}_1 \times \mathbf{e}_2|^{-1} \{|\mathbf{e}_1^* \cdot (\mathbf{q}-\mathbf{r})| + |\mathbf{e}_2^* \cdot (\mathbf{q}-\mathbf{r})|\} \, d^2\mathbf{r}/\pi = (\csc \alpha) \cdot a_1^1(\mathbf{e}_1^* \cdot \mathbf{q}, \mathbf{e}_2^* \cdot \mathbf{q}). \tag{1.14}$$

Fairthorne [5], for example, considers a triangular grid of streets.

2. Rectangular distance. For $0 \leqq t \leqq 1$ we define the function

$$f(t) = 2t^{1/2} \arcsin t^{1/2} + \tfrac{2}{3}(2+t)(1-t)^{1/2}; \tag{2.1}$$

and by direct calculation we obtain its derivative

$$f'(t) = t^{-1/2} \arcsin t^{1/2} + (1-t)^{1/2}. \tag{2.2}$$

Moreover $t^{1/2} < \arcsin t^{1/2}$ for positive t, whence

$$f''(t) = \tfrac{1}{2}t^{-1}(1-t)^{1/2} - \tfrac{1}{2}t^{-3/2} \arcsin t^{1/2} < 0 \quad \text{for } 0 < t < 1. \tag{2.3}$$

Thus $f'(t)$ is strictly decreasing on [0, 1], but is positive by (2.2); whereas $f(t)$ is strictly increasing by this remark, and is positive by (2.1). The terms in (2.1) have standard expansions about the origin, which yield a corresponding series for $f(t)$:

$$f(t) = \sum_{m=0}^{\infty} \Gamma(m-\tfrac{3}{2}) t^m / [(1-2m)\pi^{1/2}\Gamma(m+1)] = \tfrac{4}{3}F(-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}; t). \tag{2.4}$$

The symbol F in this relation is a hypergeometric function of t (Abramowitz and Stegun [1, eq. (15.1.1)]), so that (2.4) offers an analytic continuation to complex t. The coefficient of t^m is $O(m^{-7/2})$ for large m (Abramowitz and Stegun [1, eq. (6.1.47)]), so that (2.4) provides an absolutely convergent series for $|t| \leqq 1$. Hence $f(t)$, despite its definition (2.1), has no singularity at the origin.

Now we apply relation (1.9) to evaluate an auxiliary integral:

$$\begin{aligned}\iint_D |u-x|\,dx\,dy &= 2\int_{-1}^{+1}(1-x^2)^{1/2}|u-x|\,dx\\ &= 4\int_0^1 (1-x^2)^{1/2}\max(|u|,x)\,dx\\ &= 4\int_0^{|u|}(1-x^2)^{1/2}|u|\,dx + 4\int_{|u|}^1 (1-x^2)^{1/2}x\,dx\\ &= 4|u|\int_0^{\arcsin|u|}\cos^2\xi\cdot d\xi - \tfrac{4}{3}[(1-x^2)^{3/2}]_{|u|}^1\\ &= f(u^2)\quad \text{for } -1\leqq u\leqq +1.\end{aligned} \tag{2.5}$$

Then definition (1.2) and this integral imply

$$\begin{aligned}\pi a_1^1(u,v) &= \iint_D \{|u-x|+|v-y|\}\,dx\,dy\\ &= f(u^2)+f(v^2)\quad \text{for } -1\leqq u, v\leqq +1.\end{aligned} \tag{2.6}$$

Next we introduce polar coordinates (r, θ) to evaluate a second integral:

$$\begin{aligned}\iint_D |u-x|^2\,dx\,dy &= \iint_D (u^2-2ux+x^2)\,dx\,dy\\ &= \iint_D u^2\,dx\,dy - 0 + \frac{1}{2}\iint_D (x^2+y^2)\,dx\,dy\\ &= \pi u^2 + \frac{1}{2}\int_0^{2\pi} d\theta\int_0^1 r^3\,dr = \pi u^2+\frac{\pi}{4}\quad \text{for real } u.\end{aligned} \tag{2.7}$$

Also we define $Q = D\cap$ first quadrant to abbreviate our notation, and we invoke relation (1.9) to evaluate a third integral:

$$\begin{aligned}\iint_D |u-x|\,|v-y|\,dx\,dy &= \iint_Q \{|u-x|+|u+x|\}\{|v-y|+|v-y|\}\,dx\,dy\\ &= 4\iint_Q \max(|u|,x)\max(|v|,y)\,dx\,dy\\ &= 4|uv|\int_0^{|u|}dx\int_0^{|v|}dy + 2|u|\int_0^{|u|}dx\int_{v^2}^{1-x^2}d(y^2)\\ &\quad + 2|v|\int_0^{|v|}dy\int_{u^2}^{1-y^2}d(x^2) + \int_{u^2}^{1-v^2}d(x^2)\int_{v^2}^{1-x^2}d(y^2)\\ &= \tfrac{1}{2}+u^2+v^2+u^2v^2-(u^4+v^4)/6\quad \text{for } (u,v) \text{ in } D.\end{aligned} \tag{2.8}$$

Thus definition (1.2) and these integrals imply

$$\begin{aligned}\pi a_2^1(u, v) &= \iint_D \{|u-x|^2+2|u-x|\,|v-y|+|v-y|^2\}\,dx\,dy \\ &= (1+\pi/2)(1+2u^2+2v^2)+2u^2v^2-(u^4+v^4)/3 \quad \text{for } (u, v) \text{ in } D.\end{aligned} \tag{2.9}$$

The gradient of $a_1^1(u, v)$ has the form

$$\nabla a_1^1(u, v) = (2u/\pi)f'(u^2)\mathbf{i}+(2v/\pi)f'(v^2)\mathbf{j} \tag{2.10}$$

inside the disc D. This gradient has an obvious zero at the origin, but has a positive radial component elsehwere in the disc; indeed $f'(u^2)$, $f'(v^2)>0$, whence

$$(u\mathbf{i}+v\mathbf{j})\cdot\nabla a_1^1(u, v) = (2/\pi)[u^2f'(u^2)+v^2f'(v^2)]>0. \tag{2.11}$$

Also the gradient, by the symmetry of relation (2.10), has a precisely radial direction either on the coordinate axes or on the bisectors $u\pm v=0$. However, the gradient, except on these lines, has an angular component towards the nearest bisector. In proving this assertion, we may recall the symmetries (1.4) and impose the restrictions $0<v<u<1$; then

$$(-v\mathbf{i}+u\mathbf{j})\cdot\nabla a_1^1(u, v) = (2uv/\pi)[f'(v^2)-f'(u^2)] \tag{2.12}$$

is positive under these assumptions, since $f'(t)$ is decreasing for $0\leqq t\leqq 1$.

Thus the function $a_1^1(u, v)$, restricted to any circle $u^2+v^2=q^2$, assumes its minima on the coordinate axes, and assumes its maxima on the two bisectors. Specifically, the minimum value on the disc boundary is

$$a_1^1(1, 0) = 1+\frac{4}{3\pi}\simeq 1.42441 \tag{2.13}$$

while the maximum value on the disc boundary is

$$a_1^1(1/\sqrt{2}, 1/\sqrt{2}) = \left[1+\frac{10}{3\pi}\right]\Big/\sqrt{2}\simeq 1.45737. \tag{2.14}$$

Therefore the absolute maximum on the disc is (2.14), whereas the absolute minimum on the disc is

$$a_1^1(0, 0) = \frac{8}{3\pi}\simeq 0.84883. \tag{2.15}$$

We now define s and ψ by

$$s\mathbf{e}(\phi+\psi) = r\mathbf{e}(\phi+\theta)-q\mathbf{e}(\phi), \tag{2.16}$$

given any $u\mathbf{i}+v\mathbf{j}=q\mathbf{e}(\phi)$ and $x\mathbf{i}+y\mathbf{j}=r\mathbf{e}(\phi+\theta)$ in the disc D. If q and r are constants in some calculation then s and ψ are determined by θ; indeed

$$s^2 = q^2+r^2-2qr\cos\theta \tag{2.17}$$

by the law of cosines. We introduce the auxiliary function

$$g(q\mathbf{e}(\phi)) = g(u\mathbf{i}+v\mathbf{j}) = (|u|+|v|)/q = |\cos\phi|+|\sin\phi|, \tag{2.18}$$

and evaluate its rotational average

$$(2\pi)^{-1}\int_0^{2\pi} g(q\mathbf{e}(\phi))\,d\phi = \pi^{-1}\int_0^{2\pi}|\sin\phi|\,d\phi = 4/\pi. \tag{2.19}$$

We can use identity (2.19) to connect two averages (1.7):

$$\begin{aligned} b_1^1(q) &= \pi^{-1}\int_0^1 r\,dr\int_0^{2\pi} d\theta\cdot(2\pi)^{-1}\int_0^{2\pi} d\phi\cdot sg(s\mathbf{e}(\phi+\psi)) \\ &= \pi^{-1}\int_0^1 r\,dr\int_0^{2\pi} s(\theta,r)\,d\theta\cdot\frac{4}{\pi} \\ &= \frac{4}{\pi}a_1^2(u,v) = \frac{4}{\pi}b_1^2(q). \end{aligned} \tag{2.20}$$

Hence a byproduct of the later result (3.2) is the boundary average of $a_1^1(u,v)$:

$$b_1^1(1) = \frac{4}{\pi}b_1^2(1) = \frac{128}{9\pi^2} \simeq 1.44101. \tag{2.21}$$

Smeed and Jeffcoate [17], via (2.20), evaluate (2.21) in the same way, and Fairthorne [5], via (2.20), calculates the average over (u, v):

$$\iint_D a_1^1(u,v)\frac{du\,dv}{\pi} = \frac{512}{45\pi^2} \simeq 1.15281. \tag{2.22}$$

Also Haight [7] obtains (2.13), but these authors do not treat the remaining possibilities.

3. Euclidean distance. We now consider the average of the Euclidean distance: we need only calculate $b_1^2(q)$, by the rotational invariance. We first determine its extrema on the disc; our gradient analysis shows that $a_1^1(u, v)$ is strictly increasing in all radial directions, whence relation (2.20) shows that $b_1^2(q)$ is strictly increasing on $[0, 1]$. Geometric intuition might perhaps suggest using polar coordinates about the disc center, but angular integration will then produce elliptic integrals of the second kind (Fairthorne [5]). Instead we translate the origin to $(q, 0)$ and we take polar coordinates (s, ψ) about this point). Clearly if $q = 0$, so that (u, v) is at the disc center, then

$$b_1^2(0) = a_1^2(0,0) = \int_0^{2\pi} d\psi\int_0^1 s^2\,ds/\pi = \tfrac{2}{3} \simeq 0.66667; \tag{3.1}$$

while if $q = 1$, so that (u, v) is on the disc boundary, then

$$\begin{aligned} b_1^2(1) = a_1^2(1,0) &= \int_{\pi/2}^{3\pi/2} d\psi\int_0^{-2\cos\psi} s^2\,ds/\pi \\ &= -\frac{8}{3\pi}\int_{\pi/2}^{3\pi/2}\cos^3\psi\,d\psi = \frac{32}{9\pi} \simeq 1.13177. \end{aligned} \tag{3.2}$$

Hence (3.1) and (3.2), by these remarks, are the minimum and maximum on the disc.

If q is an arbitrary number in $[0, 1]$ and $(s(\psi), \psi)$ is an arbitrary point on the disc boundary, then the law of cosines asserts

$$1 = q^2 + s(\psi)^2 + 2qs(\psi)\cos\psi, \tag{3.3}$$

and the proper choice of signs implies

$$s(\psi) = -q\cos\psi + [1 - q^2\sin^2\psi)^{1/2}. \tag{3.4}$$

However, odd powers of $\cos\psi$ have zero mean, whence the required average on the disc satisfies

$$(3.5)\qquad \begin{aligned} 3\pi b_1^2(q) &= 3\pi a_1^2(q,0) \\ &= 3\int_0^{2\pi} d\psi \int_0^{s(\psi)} s^2\,ds \\ &= \int_0^{2\pi} s(\psi)^3\,d\psi \\ &= \int_0^{2\pi} [1-q^2\sin^2\psi]^{1/2}[1+3q^2-4q^2\sin^2\psi]\,d\psi \\ &= \int_0^{2\pi} [1-q^2\sin^2\psi]^{-1/2}[1+3q^2-(5q^2+3q^4)\sin^2\psi+4q^4\sin^4\psi]\,d\psi. \end{aligned}$$

However we produce no change in the value of (3.5) when we decrease its integrand by the derivative of any 2π-periodic function, while we observe

$$(3.6)\qquad \begin{aligned} &(d/d\psi)[1-q^2\sin^2\psi]^{1/2}\sin\psi\cos\psi \\ &\quad = [1-q^2\sin^2\psi]^{-1/2}[1-(2+2q^2)\sin^2\psi+3q^2\sin^4\psi] \end{aligned}$$

by direct calculation. If we subtract that multiple of (3.6) which eliminates the $\sin^4\psi$ term in (3.5), then we obtain

$$(3.7)\qquad \begin{aligned} 9\pi a_1^2(u,v) &= 9\pi a_1^2(q,0) = 9\pi b_1^2(q) \\ &= (4q^2-4)\int_0^{2\pi}[1-q^2\sin^2\psi]^{-1/2}\,d\psi + (q^2+7)\int_0^{2\pi}[1-q^2\sin^2\psi]^{1/2}\,d\psi \\ &= 16(q^2-1)K(q^2)+4(q^2+7)E(q^2) \quad \text{for } 0 \leqq q \leqq 1, \end{aligned}$$

where respectively $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kind (Abramowitz and Stegun [1, § 17.3]).

Haight [7] derives (3.2) in the same way, while Witzgall [21] obtains both (3.7) and its analogue for external (u, v). An ingenious argument via elementary functions (Whitworth [20, Exercise 696], ApSimon [2], Garwood and Tanner [6]) yields the additional average over (u, v):

$$(3.8)\qquad \iint_D a_1^2(u,v)\,du\,dv/\pi = \frac{128}{45\pi} \simeq 0.90541;$$

and the factor $4/\pi$ from (2.20) then provides the corresponding average in (2.22) (Fairthorne [5]). Various authors (Deltheil [3, pp. 114–120], Hammersley [8], Watson [19], Schweitzer [15], Wyler [23]) calculate higher moments for two random points; moreover the first two consider higher dimensions, while the last two generalize the external average, and Hammersley [9] cites a biological application of such results. We shall not repeat these calculations, but, recalling the integral (2.7), we obtain the desired second moment:

$$(3.9)\qquad \begin{aligned} \pi b_2^2(q) = \pi a_2^2(u,v) &= \iint_D \{|u-x|^2+|v-y|^2\}\,dx\,dy \\ &= (\pi/2)(1+2u^2+2v^2) = (\pi/2)(1+2q^2) \quad \text{for } 0 \leqq q. \end{aligned}$$

Some partially numerical results of Karp, McKellar and Wong [12] suggest an almost exact proportionality among the averages $a_1^p(u, v)$. Hence, on the disc D, we consider the ratio $a_1^1(u, v)/a_1^2(u, v)$, or equivalently, by (2.20), we study the ratio $a_1^1(u, v)/b_1^1(q)$. The rotational minimum, average, and maximum of $a_1^1(u, v)$ satisfy respectively

$$\begin{aligned}\pi a_1^1(q, 0) &= f(q^2)+f(0)\\ &= \tfrac{8}{3}+\sum_{m=1}^{\infty}\Gamma(m-\tfrac{3}{2})q^{2m}/[(1-2m)\pi^{1/2}\Gamma(m+1)],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\pi b_1^1(q) &= (2\pi)^{-1}\int_0^{2\pi}[f(q^2\cos^2\phi)+f(q^2\sin^2\phi)\,d\phi\\ &= \pi^{-1}\sum_{m=0}^{\infty}\Gamma(m-\tfrac{3}{2})q^{2m}\int_0^{2\pi}\sin^{2m}\phi\,d\phi/[(1-2m)\pi^{1/2}\Gamma(m+1)]\\ &= -\sum_{m=0}^{\infty}\Gamma(m-\tfrac{1}{2})\Gamma(m-\tfrac{3}{2})q^{2m}/[\pi\Gamma(m+1)^2],\end{aligned} \tag{3.11}$$

$$\begin{aligned}\pi a_1^1(q/\sqrt{2}, q/\sqrt{2}) &= 2f(q^2/2)\\ &= \sum_{m=0}^{\infty}2\Gamma(m-\tfrac{3}{2})q^{2m}/[(1-2m)2^m\pi^{1/2}\Gamma(m+1)],\end{aligned} \tag{3.12}$$

by (2.4) and (2.6). All three series, for $|q|\leqq 1$, are absolutely convergent; the first two terms in each expansion are respectively identical; only these terms in each expansion are ever positive. Thus, in particular,

$$\pi b_1^1(q)=\tfrac{8}{3}+2q^2-\cdots; \tag{3.13}$$

and, by monotonicity,

$$\tfrac{8}{3}\leqq \pi b_1^1(q), \qquad (d/dq^2)\pi b_1^1(q)\leqq 2. \tag{3.14}$$

Not only do the three preceding functions, by our gradient analysis, take positive values in the stated order, but also their consecutive differences, in the same order, are power series of positive terms. This follows respectively from the inequalities

$$2\Gamma(m+\tfrac{1}{2})<\Gamma(\tfrac{1}{2})\Gamma(m+1)<2^m\Gamma(m+\tfrac{1}{2}) \tag{3.15}$$

for $m=2, 3, \cdots$, which follow immediately by induction from the case $m=1$. Therefore

$$\begin{aligned}&\pi(b_1^1(q)-a_1^1(q, 0)]=q^4\times\text{strictly increasing function},\\ &\pi[a_1^1(q/\sqrt{2}, q/\sqrt{2})-b_1^1(q)]=q^4\times\text{strictly increasing function}.\end{aligned} \tag{3.16}$$

However $(d/dq)\log[q^4/b_1^1(q)]$ is strictly positive for $0\leqq q<1$, since

$$\pi q(d/dq)b_1^1(q)=2q^2(d/dq^2)\pi b_1^1(q)\leqq 4q^2\leqq 4<\tfrac{32}{3}\leqq 4\pi b_1^1(q) \tag{3.17}$$

by (3.14). Hence the differences (3.16), even multiplied by $1/(\pi b_1^1(q))$, remain strictly increasing functions on $[0, 1]$, and assume their maxima on the disc boundary. However the ratio $a_1^1(u, v)/b_1^1(q)$ is unity at the origin, whereas

$$a_1^1(1, 0)/b_1^1(1)=3\pi(3\pi+4)/128\simeq 0.98848, \tag{3.18}$$

$$a_1^1(1/\sqrt{2}, 1/\sqrt{2})/b_1^1(1)=3\pi(3\pi+10)/(128\sqrt{2})\simeq 1.01135. \tag{3.19}$$

Clearly (3.18) and (3.19), by these arguments, are the minimum and maximum of $a_1^1(u, v)/b_1^1(q)$, whence

$$a_1^1(1, 0)/a_1^2(1, 0) = 3(3\pi + 4)/32 \simeq 1.25857, \tag{3.20}$$

$$a_1^1(1/\sqrt{2}, 1/\sqrt{2})/b_1^1(1) = 3(3\pi + 10)/(32\sqrt{2}) \simeq 1.28769, \tag{3.21}$$

by (2.20), are the minimum and maximum of $a_1^1(u, v)/a_1^2(u, v)$. Thus the ratio is indeed very nearly constant.

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